Properties of Conic State Transition Matrices and Associated Time Partials

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Transition matrices for conic trajectories are determined for a variety of cutoff constraints, each of which terminates the perturbed trajectory at a different point. The relationships between these transition matrices, by means of their corresponding time partials, are developed. Also, several ways of verifying state transition matrices and their associated time partials are presented including, in some cases, the use of the transition matrix to update the state vector.

Introduction

FREQUENTLY an error analysis must be performed when selecting a trajectory for a space mission. Error analyses involve updating individual state errors (in position and velocity) or an ensemble of state errors in the form of a covariance matrix through a state transition matrix. The most common transition matrix is the time transition matrix,[1] which is defined by holding the flight time along the perturbed trajectory equal to that along the nominal trajectory. There are many other transition matrices that can be defined, each valid for a specified "cutoff condition" which serves to define a unique terminal point on the perturbed trajectory. The nature of the space mission dictates which transition matrices are of interest. For a trajectory that impacts on the surface of a planet, the transition matrix, which terminates the pertubed trajectory on the surface of the planet, would be of primary interest. A time partial which defines the change in time of flight is associated with each transition matrix (except for the time transition matrix).

The calculation of conic transition matrices for two-body trajectories is a fairly straightforward, but lengthy, exercise in differentiation. Assuming that one transition matrix is available, this paper presents the interrelationships between different transition matrices and between the time partials and the transition matrices. The equation relating the time transition matrix to other transition matrices has been published.[2] This paper also presents several ways of verifying the transition matrices and their corresponding time partials. In addition, it is shown how several state transition matrices (including the time transition matrix) can be used to update the state vector as well as perturbations in the state vector.

The 6×6 state transition matrix, Φ, propagates the initial state error δX in position δR and velocity δV [$\delta X=(\delta R,\delta V)$] into the final state error δX_f.

$$\delta X_f=\Phi\delta X \tag{1}$$

The corresponding six-dimensional time partial vector, $\delta t_f/\delta X$, when dotted with the initial state error, yields the final time of flight error δt_f

$$\delta t_f=\frac{\delta t_f}{\delta X}\delta X \tag{2}$$

Received Aug. 30, 1982; revision received March 12, 1983.

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The transition matrix can be derived by differentiating the final state vector (R_f, V_f), expressed as a function of the initial state vector (R, V) and the desired scalar cutoff condition, holding the derivative of that cutoff condition equal to zero. In a similar fashion, the time of flight, expressed as a function of the same parameters, can be differentiated to yield the time partials. Any previously obtained transition matrix can be used to obtain any other transition matrix by a simple matrix calculation (with the exception that the time transition matrix cannot be obtained from any other transition matrix). Any time partial can be obtained using the time transition matrix.

Different Cutoff Conditions

Each cutoff condition imposes a scalar constraint on the perturbed trajectory which uniquely determines the final point on the trajectory. There are two general categories of cutoff conditions. One category, referred to as "traverse" constraints, ensures that the perturbed trajectory traverses the same specified condition between the initial and final points as was traversed on the nominal trajectory. Examples are time, central angle, velocity magnitude, and radial distance (see Fig. 1 for several examples). α will be used to represent a traverse constraint.

"Terminal" constraints is another category of cutoff conditions which terminate the perturbed trajectory on some specified condition. All of the cutoff conditions used as traverse constraints, except for time, also can be used as terminal constraints. Several examples are shown in Fig. 2. β will be used to represent a terminal constraint.

Both categories of cutoff conditions can be mathematically formulated by introducing a scalar linear relationship between the initial and final state errors and two constraint vectors, K and L.[2] These constraint vectors are related to the perturbations in the state vectors through the scalar equation

$$K^T\delta X_f+L^T\delta X=0 \tag{3}$$

K and L represent the partial derivatives of the cutoff conditions with respect to the final and initial state vectors. Traverse constraints require that both K and L have some nonzero elements, whereas terminal constraints require that L is identically equal to zero.

Equation (1) can be inserted into Eq. (3) with the result

$$(K^T\Phi+L^T)\delta X=0 \tag{4}$$

Since δX is arbitrary, Eq. (4) yields the expressions

$$K^T\Phi_\alpha+L^T=0^T, \qquad K^T\Phi_\beta=0^T \tag{5}$$

where the subscript indicates the nature of the transition matrix. Equation (5) provides a numerical check on the validity of specific transition matrices.

Determination of Traverse Constraint Vectors

The central angle is an example of a traverse cutoff condition, which involves perturbations in both δX_f and δX. This condition ensures that the central angles traversed on the nominal and perturbed trajectories are identical (see Fig. 1). The central angle $\Delta\theta$ is defined by

$$\Delta\theta = \cos^{-1}(R^T R_f / r r_f) \tag{6}$$

where the lower-case letters are used to represent the magnitude of the vector indicated by the corresponding uppercase letter. Taking the total differential of the central angle and setting the result equal to zero leads to the constraint vectors K and L

$$K^T = \left[\frac{R^T}{r}\left(\frac{I}{r_f} - \frac{R_f R_f^T}{r_f^3}\right), 0^T\right]$$

$$L^T = \left[\frac{R_f^T}{r_f}\left(\frac{I}{r} - \frac{R R^T}{r^3}\right), 0^T\right] \tag{7}$$

where I is an identity matrix. The differential relationship between time and range angle θ at either end of the trajectory is given by

$$\frac{\partial t}{\partial \theta} = \frac{r}{v_\theta} \tag{8}$$

This partial is required to relate δt to $\delta\theta$ since the state vector (involving derivatives with respect to time) is updated based on time and not on central angle.

Flight through a specified radical distance, Δr, is another example of a traverse constraint.

$$\Delta r = r - r_f \tag{9}$$

Taking the total differential of Δr and setting the result equal to zero results in

$$R^T \delta R / r = R_f^T \delta R_f / r_f \tag{10}$$

leading to the constraint vectors

$$K^T = (R_f^T / r_f, 0^T), \qquad L^T = (-R^T / r, 0^T) \tag{11}$$

The differential relationship between position r and time is

$$\frac{\partial t}{\partial r} = \frac{1}{v_r} \tag{12}$$

A traverse based on velocity, which leads to the constraint vectors

$$K^T = (0^T, V_f^T / v_f), \qquad L^T = (0^T, -V^T / v) \tag{13}$$

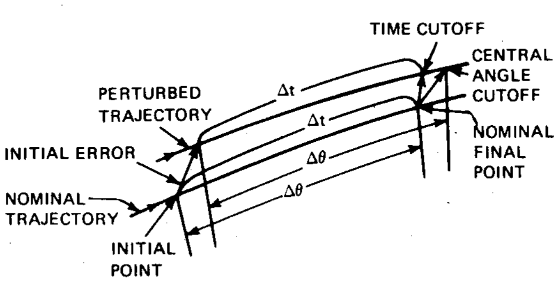


Fig. 1 Examples of traverse constraints.

is an example similar to radial distance. The differential relationship between velocity v and time is given by

$$\frac{\partial t}{\partial v} = \frac{v}{A^T V} \tag{14}$$

where A is the acceleration vector.

Determination of Terminal Constraint Vectors

The easiest category of constraint vectors to derive for terminal constraints involves stopping the perturbed trajectory on a plane defined in either position or velocity space by a vector N perpendicular to the plane. In this case, constraints in position space ($N \cdot \delta R_f = 0$) and in velocity space ($N \cdot \delta V_f = 0$) lead to the following constraint vectors:

$K^T = (R_f^T, 0^T)$ = final radial distance (also referred to as altitude cutoff)

$K^T = \{[(R_f \times V_f) \times R_f]^T, 0^T\}$ = central angle (i.e., cutoff above target)

$K^T = \{0^T, [(R_f \times V_f) \times V_f]^T\}$ = independent flight-path angle (i.e., final perturbation in velocity colinear with final velocity vector)

$K^T = (0^T, R_f^T)$ = final radial component of velocity

$K^T = (0^T, V_f^T)$ = final velocity (i.e., no error in velocity magnitude)

Two practical applications of the above terminal constraints are using the altitude constraint when the perturbed trajectory is assumed to impact the surface of the Earth, and using the central angle cutoff when the perturbed trajectory is assumed to terminate directly above the target (which might be the case if a mapping device was employed).

Flight-path angle is an example of a cutoff condition which does not terminate on a position or velocity plane. Differentiating this angle (equal to the arccos of the dot product of the unit position vector and the unit velocity vector when measured with respect to the vertical) and setting it equal to zero results in the constraint vector K

$$K^T = \left(V_f^T - \frac{R_f V_f R_f^T}{r_f^2}, R_f^T - \frac{R_f V_f V_f^T}{v_f^2}\right) \tag{15}$$

The terminal cutoff condition that minimizes the position error at the target for no initial error in time is obtained by minimizing the final position error with respect to the time error δt_f

$$\frac{d\,|\delta R_f|}{d(\delta t_f)} = 0 \tag{16}$$

leading to

$$\delta R_f^T \frac{d(\delta R_f)}{d(\delta t_f)} = 0 \tag{17}$$

Since

$$\frac{d(\delta R_f)}{d(\delta t_f)} = V_f - V_{tf} \tag{18}$$

where V_{tf} represents the velocity vector of a target, the constraint vector K is given by

$$K^T = (V_f^T - V_{tf}^T, 0^T) \tag{19}$$

All of the above terminal constraint cases involve null L vectors.

It should be noted that the terminal constaint vector K can be obtained frequently from the K vector for the corresponding traverse constraint vector. (For terminal constraints, the length of K is arbitrary.) For example, the K vector in Eq. (11) for a radial distance traverse is the K vector for a terminal radial distance.

Propagation of the State Vector

The easiest way to establish how to update the state vector with the state transition matrix for a traverse cutoff condition is to consider state errors (which are arbitrary) generated by multiplying the dynamical state (composed of velocity and accelertion) by a perturbation in time. In this case, the perturbed trajectory coincides with the nominal trajectory and the perturbations at each end of the trajectory are expressed as follows

$$\delta X = D\delta t = D\frac{\partial t}{\partial \alpha}\delta\alpha$$

$$\delta X_f = D_f \delta t_f = D_f \frac{\partial t_f}{\partial \alpha_f}\delta\alpha_f \tag{20}$$

where

$$D_f = \begin{bmatrix} V_f \\ A_f \end{bmatrix}, \quad D = \begin{bmatrix} V \\ A \end{bmatrix}$$

Considering the definition of Φ in Eq. (1) and noting that the perturbation in the cutoff condition α must be equal at both ends of the trajectory (since $\Delta\alpha$ is held invariant), the equation for updating the dynamical state vector with any traverse transition matrix is

$$\Phi_\alpha D \frac{\partial t}{\partial \alpha} = D_f \frac{\partial t_f}{\partial \alpha_f} \tag{21}$$

Note that updating the transition matrix by an integer number of periods will change the transition matrix while leading to the same prediction of the dynamical state. This indicates that, due to the multiplicity of solutions, it is impossible to derive the transition matrix from just knowledge of the initial and final dynamical state vectors.

The acceleration can be eliminated in the case of the two-body problem in terms of the position (i.e., $A = -\mu R/r^3$, where μ is the gravitation constant), in which case the propagation of the state can be expressed as

$$\Phi_\alpha \begin{bmatrix} V \\ -\mu R/r^3 \end{bmatrix} \frac{\partial t}{\partial \alpha} = \begin{bmatrix} V_f \\ -\mu R_f/r_f^3 \end{bmatrix} \frac{\partial t_f}{\partial \alpha_f} \tag{22}$$

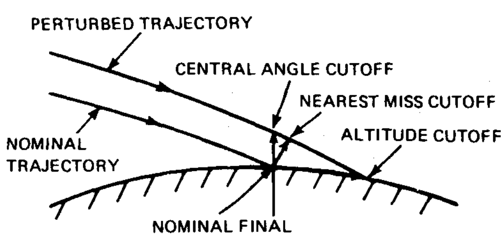


Fig. 2 Examples of terminal constraints.

where r_f is found by taking the magnitude of the calculated acceleration vector

$$r_f = (\mu/a_f)^{1/2} \tag{23}$$

For the case of time cutoff, the partial $\partial t/\partial\alpha$ at each end of the trajectory is 1 and Eq. (21) reduces to

$$\Phi_t D = D_f \tag{24}$$

where Φ_t is the time transition matrix.

Since Φ_t is a symplectic matrix,[3] Eq. (24) can be written in a somewhat more standard form

$$(\Phi_t^{-1})^T \begin{bmatrix} \mu R/r^3 \\ V \end{bmatrix} = \begin{bmatrix} \mu R_f/r_f^3 \\ V_f \end{bmatrix} \tag{25}$$

The first derivative of the acceleration is related to the velocity according to

$$\dot{A} = GV \tag{26}$$

allowing the propagation of the time derivative of the dynamical state $(A, \dot{A})$ to be obtained by inserting Eq. (26) into Eq. (24)†

$$\begin{bmatrix} 0 & I \\ G_f & 0 \end{bmatrix} \Phi_t \begin{bmatrix} 0 & G^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} A \\ \dot{A} \end{bmatrix} = \begin{bmatrix} A_f \\ \dot{A}_f \end{bmatrix} \tag{27}$$

The gravity gradient matrix G is given by

$$G = -\frac{\mu}{r^3}\left(I - \frac{3RR^T}{r^2}\right) \tag{28}$$

In the case of the traverse constraint of central angle

$$\frac{\partial t_f}{\partial \theta_f} = \frac{r_f}{v_{\theta f}}, \qquad \frac{\partial t}{\partial \theta} = \frac{r}{v_\theta} \tag{29}$$

Combining Eqs. (21) and (29) and using the principle of conservation of angular momentum results in the updating of the dynamical state vector with the traverse central angle transition matrix Φ_θ

$$\Phi_\theta r^2 D = r_f^2 D_f \tag{30}$$

The reason why traverse matrices allow the propagation of the state vector is that the final state error always depends on where the initial state error is defined on the perturbed trajectory. For terminal constraints, the final error is independent of where the initial error is defined on the perturbed trajectory. In this case, multiplying the initial dynamical state by the terminal transition matrix results in a null vector [proved in Eq. (45)].

The propagation of the state error in the form δX is one of six possible ways of relating initial and final errors based on permutating the four quantities: δR, δV, δR_f, and δV_f.

$$\begin{matrix} \delta R, \delta V, & \delta R, \delta R_f, & \delta R, \delta V_f \\ \delta R_f, \delta V, & \delta R_f, \delta V_f, & \delta V, \delta V_f \end{matrix} \tag{31}$$

†Originally derived by Stan Shepperd, The Charles Stark Draper Laboratory, Inc.

The corresponding transformation matrices are

$$\begin{bmatrix} \delta R_f \\ \delta V_f \end{bmatrix} = A \begin{bmatrix} \delta R \\ \delta V \end{bmatrix}, \begin{bmatrix} \delta V \\ \delta V_f \end{bmatrix} = B \begin{bmatrix} \delta R \\ \delta R_f \end{bmatrix}$$

$$\begin{bmatrix} \delta R_f \\ \delta V \end{bmatrix} = C \begin{bmatrix} \delta R \\ \delta V_f \end{bmatrix}, \begin{bmatrix} \delta R \\ \delta V_f \end{bmatrix} = D \begin{bmatrix} \delta R_f \\ \delta V \end{bmatrix}$$

$$\begin{bmatrix} \delta R \\ \delta V \end{bmatrix} = E \begin{bmatrix} \delta R_f \\ \delta V_f \end{bmatrix}, \begin{bmatrix} \delta R \\ \delta R_f \end{bmatrix} = F \begin{bmatrix} \delta V \\ \delta V_f \end{bmatrix} \tag{32}$$

In the case of time cutoff, replacing δR and δV with V and A, respectively, in Eq. (32), allows the matrices A, B, C, D, E, and F to propagate the state vector.

One matrix worth special study in Eq. (32), when they are all based on the time cutoff condition, is B which can be considered as the Lambert matrix since it defines velocities as functions of position. By setting δR_f equal to zero, the submatrix B_1 defines the fixed time of arrival guidance matrix ($\partial V/\partial R$, historically referred to as the "Q" matrix). Since the matrices in Eq. (32) are all interrelated, any matrix can be used to derive any other matrix. For instance, using the A matrix it can be shown that

$$\delta V_f = (A_3 - A_4 A_2^{-1} A_1)\delta R + A_4 A_2^{-1} \delta R_f$$

$$\delta V = A_2^{-1}(\delta R_f - A_1 \delta R) \tag{33}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

Since A $(=\Phi_t)$ is symplectic, it can be shown that

$$A_2^{-1} = -A_3^T + A_1^T (A_2^T)^{-1} A_4^T \tag{34}$$

which allows B to be expressed as

$$B = \begin{bmatrix} -A_2^{-1}A_1 & A_2^{-1} \\ -(A_2^{-1})^T & A_4 A_2^{-1} \end{bmatrix} \tag{35}$$

B is a function of A_1, A_2, and A_4 since there are only three independent submatrices in the symplectic A matrix. The Lambert matrix propagates the state in the following form:

$$B \begin{bmatrix} V \\ V_f \end{bmatrix} = \begin{bmatrix} A \\ A_f \end{bmatrix} \tag{36}$$

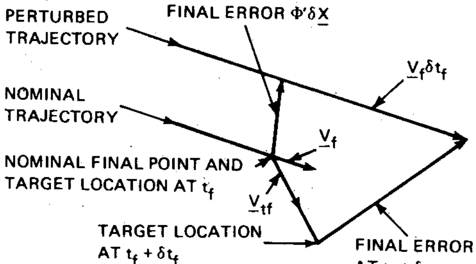


Fig. 3 Geometry of the terminal error.

Time Partial

The time partial associated with any transition matrix can be derived by linearizing the motion in the vicinity of the terminal point (see Fig. 3). To generalize the problem, introduce a moving target at the final point by introducing its dynamical state D_{tf}. The desired terminal error δX_f can be defined in terms of any transition matrix Φ' and the dynamical states of the nominal and target trajectories propagated through a change in flight time, δt_f, measured from the point represented by Φ'

$$\delta X_f = \Phi' \delta X + D_f \delta t_f - D_{tf} \delta t_f \tag{37}$$

where

$$D_{tf} = \begin{bmatrix} V_{tf} \\ A_{tf} \end{bmatrix}$$ velocity and acceleration of the target at the terminal point

Equation (3) can be combined with Eq. (37) and solved for the change in flight time

$$\delta t_f = \frac{(-K^T \Phi' - L^T)\delta X}{K^T \Delta D_f} \tag{38}$$

where

$$\Delta D_f = D_f - D_t$$

The time partial follows from Eq. (38)

$$\left.\frac{\partial t_f}{\partial X}\right|_{\Phi'}^{\Phi} = \frac{-\Phi'^T K - L}{K^T \Delta D_f} \tag{39}$$

The subscript and superscript on the time partial indicates that the time extends from the cutoff point represented by Φ' to the cutoff point represented by Φ (specified by the constraint vecotors K, L), respectively. The time partial vector in Eq. (39) is composed of two parts: $\Phi'^T K$ which extends from the Φ' point to the point defined by the terminal cutoff condition K, and L which extends from the point defined by the terminal cutoff condition K to the point represented by the K,L cutoff condition.

Since the transition matrix postmultiplied by the initial dynamical state updates the dynamical state in traverse cutoff cases, it is interesting to examine the dot product of the time partial vector with the initial dynamical state. Multiplying the time partial, Eq. (39), by the initial dynamical state (neglecting the target state) results in

$$\left.\frac{\partial t_f}{\partial X}\right|_{\Phi'}^{\Phi} D = \frac{-K^T \Phi' D - L^T D}{K^T D_f} = \frac{\delta t_f}{\delta t} \tag{40}$$

This equation can be simplified by noting that the constraint relationship, Eq. (3), in the case of a traverse cutoff, can be divided by $\delta\alpha = (\partial\alpha/\partial t)\delta t$ allowing the dynamical state to be introduced, resulting in

$$\frac{L^T D}{K^T D_f} = -\left(\frac{\partial t_f}{\partial \alpha_f} \Big/ \frac{\partial t}{\partial \alpha}\right) \tag{41}$$

Table 1 summarizes the results of dotting the time partial with the initial dynamical state for various input-output conditions based on Eqs. (21), (40), and (41).

Interrelation Between Transition matrices

The change in flight time δt_f, Eq. (38), can be inserted into Eq. (37) with the result

$$\delta X_f = \left[\Phi' - \frac{\Delta D_f (K^T \Phi' + L^T)}{K^T \Delta D_f}\right] \delta X \tag{42}$$

The transition matrix Φ for the constraint vectors K and L is thereby given by

$$\Phi=\left[\Phi'-\frac{\Delta D_f(K^T\Phi'+L^T)}{K^T\Delta D_f}\right]$$

$$=\left[I-\frac{\Delta D_f K^T}{K^T\Delta D_f}\right]\Phi'-\frac{\Delta D_f L^T}{K^T\Delta D_f} \tag{43}$$

The matrix in the bottom row of Eq. (43) is an idempotent matrix (i.e., a square matrix M which has the property $M^2=M$). If the initial constraint vector L is a null vector, Eq. (42) reduces to

$$\delta X_f=\left[I-\frac{\Delta D_f K^T}{K^T\Delta D_f}\right]\Phi'\delta X=\Phi_t\Phi'\delta X=\Phi_\beta\delta X \tag{44}$$

where Φ_β is the transition matrix based on the terminal constraint K. Hence, an idempotent matrix relates a terminal constraint matrix to any other transition matrix. It is not possible to invert Eq. (44) and obtain Φ' from Φ_β since Φ_t is a singular matrix.

For a terminal constraint case (L equals a null vector) with no target trajectory, the transition matrix Φ_β times the initial dynamic state vector is the null vector. This is apparent from inserting the definition of the terminal constraint transition matrix in terms of the time transition matrix into $\Phi_\beta D$.

$$\Phi_\beta D=\left[I-\frac{D_f K^T}{K^T D_f}\right]\Phi_t D=\left[I-\frac{D_f K^T}{K^T D_f}\right]D_f=0 \tag{45}$$

To verify that any transition matrix can be used in Eq. (43) to yield a desired terminal or traverse cutoff transition matrix (except when it is the time transition matrix), an intermediate transition matrix based on a K_x, L_x constraint can be inserted into Eq. (43)

$$\Phi_\alpha=\left[I-\frac{\Delta D_f K^T}{K^T\Delta D_f}\right]\left[\Phi'-\frac{\Delta D_f K_x^T\Phi'}{K_x^T\Delta D_f}-\frac{\Delta D_f L_x^T}{K_x^T\Delta D_f}\right]-\frac{\Delta D_f L^T}{K^T\Delta D_f}$$

$$=\left[I-\frac{\Delta D_f K^T}{K^T\Delta D_f}\right]\Phi'-\frac{\Delta D_f L^T}{K^T\Delta D_f} \tag{46}$$

The time transition matrix can be obtained from any other transition matrix if the corresponding time partial is available.

Table 1 Time partials for various cutoff conditions

Input ϕ'	Output Φ	$\left.\dfrac{\partial t_f}{\partial X}\right/_{\Phi'}^{\Phi} D$
Φ_t'	Φ_β	-1
Φ_t'	Φ_α	$-1+\dfrac{\partial t_f}{\partial \alpha_f}\Big/\dfrac{\partial t}{\partial \alpha}$
Φ_β'	Φ_β	0
Φ_β'	Φ_α	$\dfrac{\partial t_f}{\partial \alpha_f}\Big/\dfrac{\partial t}{\partial \alpha}$
Φ_α'	Φ_β	$-\dfrac{\partial t_f'}{\partial \alpha_f'}\Big/\dfrac{\partial t'}{\partial \alpha'}$
Φ_α'	Φ_α	$-\dfrac{\partial t_f'}{\partial \alpha_f'}\Big/\dfrac{\partial t'}{\partial \alpha'}+\dfrac{\partial t_f}{\partial \alpha_f}\Big/\dfrac{\partial t}{\partial \alpha}$

$$\Phi_t=\Phi'+D_f\left.\frac{\partial t_f^T}{\partial X}\right|_{\Phi'}^{\Phi_t} \tag{47}$$

If the time partial associated with the points Φ and Φ' is available, then

$$\Phi=\Phi'+D_f\left.\frac{\partial t_f^T}{\partial X}\right|_{\Phi'}^{\Phi} \tag{48}$$

Table 2 contains the time partials for various cutoff conditions for an initial error in radial position for the two-body trajectory defined in an inertial coordinate frame by

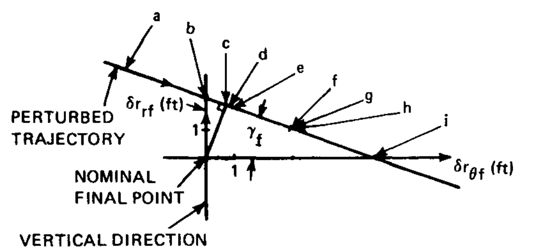
$$R=(0,\ 0,\ 21304944)\ \text{ft}$$

$$V=(21315.504,\ 0,\ 7323.0274)\ \text{ft/s} \tag{49}$$

with a flight time of 1537.0326 s ($\mu=1.40764668\times10^{16}$ ft^3/s^2). Figures 4 and 5 show the position errors and the velocity errors, respectively, for an initial error in the radial direction of 1 ft.

Additional Checks on the Transition Matrices

Additional checks can be easily performed on the transition matrices after rotating them into a local vertical coordinate frame at each end of the trajectory. All transition matrices then will share the same row of partials involving the final horizontal component of velocity. This is because introducing a differential element of flight time (δt_f) does not, to the first



a TIME TRAVERSE
b TERMINAL CENTRAL ANGLE AND CENTRAL ANGLE TRAVERSE
c TERMINAL NEAREST MISS
d TERMINAL INDEPENDENT FLIGHT PATH ANGLE
e TERMINAL RADIAL VELOCITY
f RADIAL DISTANCE TRAVERSE
g TERMINAL VELOCITY AND VELOCITY TRAVERSE
h TERMINAL FLIGHT PATH ANGLE
i TERMINAL RADIAL DISTANCE

Fig. 4 Different cutoff points in position space along a perturbed trajectory.

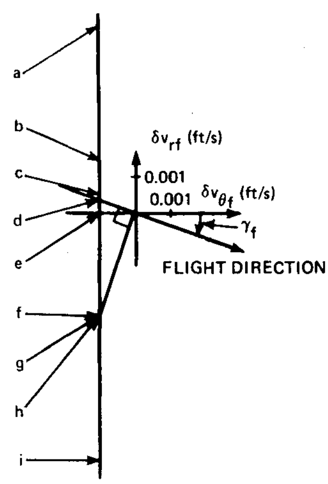


Fig. 5 Different cutoff points in velocity space along a perturbed trajectory.

Table 2 Time partials for various cutoff conditions

Cutoff condition	Time partial, $\times 10^{-4}$ $\partial t_f/\partial r_r$, s/ft
Time	0
Terminal central angle and central angle traverse	1.304
Terminal nearest miss	1.603
Terminal independent flight-path angle	1.633
Terminal radial velocity	1.753
Radial distance traverse	2.730
Terminal velocity and velocity traverse	2.743
Terminal flight-path angle	2.764
Terminal altitude	4.072

order, result in any change in the horizontal component of velocity since the acceleration vector (i.e., the gravity vector) is in the vertical direction. Hence, the minimum velocity error will always occur when the radial velocity error is zero.

Another check is that the in-plane partial of the final horizontal velocity with respect to the initial horizontal component of the position vector equals the out-of-plane partial of final velocity with respect to initial position

$$\frac{\partial v_{\theta f}}{\partial r_\theta} = \frac{\partial v_{zf}}{\partial r_z} \tag{50}$$

The rotation of the position vector through $\delta r_\theta / r$ or $\delta r_z / r$, while maintaining the direction of the velocity vector, results in the same induced velocity error in the horizontal and out-of-plane directions. Due to the conservation of angular momentum, this induced velocity error has the same effect on the final velocity in both the in-plane and out-of-plane directions.

Some other simple checks on transition matrices, when expressed in the local vertical coordinate frames, are:

1) Terminal central angle—downrange position partials all zero.

2) Terminal radial distance—radial position partials all zero.

3) Terminal velocity—velocity partials define an error which is perpendicular to the flight direction.

4) Terminal independent flight-path angle—velocity partials define an error which is parallel to the flight direction.

5) Terminal nearest miss—position partials define an error which is perpendicular to the flight direction.

6) Terminal radial velocity—radial velocity partials all zero.

7) Traverse central angle—downrange position partials all zero except $\partial r_{\theta f}/\partial r$ which equals r_f/r.

8) Traverse radial distance—radial position partials all zero except $\partial r_{rf}/\partial r$ which equals 1.

9) Traverse central angle—the following two partials are defined by

$$\frac{\partial v_{rf}}{\partial v_r} = \cos\theta, \qquad \frac{\partial v_{\theta f}}{\partial v_r} = -\sin\theta \tag{51}$$

The partials in Eq. (51) show that an initial radial velocity error will remain constant in inertial space whenever the final error on the perturbed trajectory occurs radially above the final point.

Conclusions

Any conic transition matrix and its corresponding time partial can be used to obtain any other transition matrix (except time cutoff) and its corresponding time partial. An idempotent matrix relates a terminal constraint transition matrix to any other transition matrix. Traverse cutoff transition matrices allow the dynamical state vector of the trajectory to be updated. A terminal cutoff transition matrix times the initial dynamical state vector equals a null vector. A time partial times the initial dynamical state can be expressed as a simple function of the partial derivatives of time with respect to the cutoff condition at both ends of the trajectory. All transition matrices (except for time cutoff) are simply related to their constraint vectors. All transition matrices, when expressed in a local vertical coordinate frame at the final point, share an identical row of partials. All transition matrices, when expressed in local vertical coordinates at both ends of the trajectory, have one in-plane partial equal to an out-of-plane partial.

Acknowledgment

This report was prepared by The Charles Stark Draper Laboratory, Inc., under Contract F04704-81-C-0044 with the Air Force Ballistic Missile Office at Norton Air Force Base, California.

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